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Delay can stabilize: Love affairs dynamics

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ABSTRACT

We discuss two models of interpersonal interactions with delay. The first model is linear, and allows the presentation of a rigorous mathematical analysis of stability, while the second is nonlinear and a typical local stability analysis is thus performed. The linear model is a direct extension of the classic Strogatz model. On the other hand, as interpersonal relations are nonlinear dynamical processes, the nonlinear model should better reflect real interactions. Both models involve immediate reaction on partner's state and a correction of the reaction after some time.

The models we discuss belong to the class of two-variable systems with one delay for which appropriate delay stabilizes an unstable steady state. We formulate a theorem and prove that stabilization takes place in our case. We conclude that considerable (meaning large enough, but not too large) values of time delay involved in the model can stabilize love affairs dynamics.

1. Introduction

1.1. Delay in human emotions

In psychology of emotion, this formation of emotion is described by so-called differential emotions theory (DET) in which forming an emotion can be modeled by a dynamics of a system undergoing neurohormonal, motoric and experiential processes [1]. The core assumption in DET is that small set of emotions are primary and independent: joy, interest, sadness, fear, surprise and disgust [2]. These are independent because they achieve consciousness rapidly and automatically, and influence subsequent perception and cognition. The second important claim is that these emotions are discrete (associated with a specific neuromuscular pattern of facial movements) and distinguishable. Individual emotions also undergo interactions with other emotions in order to form emotion patterns that stabilize over repetitions and time and refer to compound emotions. Thus, single emotions are both the product and the subject of system organization. The systems are self-organizing in the sense that recursive interactions among component processes generate emergent properties.

There are two groups of compound emotions in terms of time necessary for the system to form them. First group is emotions that can be both close to immediate, or delayed: anxiety, anger, irritability, guilt, feeling overwhelmed, grief, hopelessness. However, there is also a group of emotions that are always delayed because it takes long for the system to reach them: feeling abandoned, resentment, feeling of alienation, withdrawal, numbness, depression.

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1.2. Modeling of interpersonal relationships

Understanding the dynamics of marriage or other close personal relationships is a goal of various sociological and socio-psychological studies, including studies based on the mathematical sociology approach. Numerous papers presenting dynamical systems describing such relationships can be found, cf. e.g. [3,4,5,6,7,8,9,10,11,12,13,14,15,16,17]. Furthermore, in 1994 Gottman et al. published the text-book focused on that problem, where they used mathematical modeling in the description of different types of marriage and divorce prediction. Typically, the discrete dynamical systems approach is used in this type of modeling, focusing on nonlinear dynamics of interactions. However, continuous dynamical systems can be used as well, not excluding linear ODEs systems.

Models describing social interactions between agents using systems of ordinary differential equations (ODEs), especially derived from the game theory, are quite common nowadays, cf. the classic text-book of [18] or article by Axelrod [19] and the references therein. Also models describing human interactions as a process of responsive behavior has been considered by Platkowski and Poleszczuk [20]. In this section, we would like to discuss a problem of modeling personal interactions with delay, and provide argumentation for the delay to be incorporated into the modeling, while in Section 2 we introduce the model itself, and describe the meaning of variables to be used.

Models with time delays are quite often used to describe population dynamics, cf. e.g. a text-book by Gopalsamy [21]. It is obvious that time delays can be always observed in natural environment: it takes some time for the signal to travel from one cell to the other, the process of transcription of a protein also takes some time, cf. e.g. [22] or [23].

We find it reasonable and necessary to combine the philosophy standing over the approaches mentioned above. Constructing a mathematical model we always have to simplify described process. This implies that we need to take a decision which parts of the described phenomena can be neglected and which not. Moreover, we need to choose a simplification level: either to describe a given process more precisely which would lead to a complex system of many equations with plenty of parameters that hardly can be analyzed by mathematical tools, or to simplify the description and reduce the number of variables/equations in order to give ourselves a chance for mathematical analysis and parameter identification. In the latter case it is sometimes useful to introduce time delays into the system in order to reflect that some processes take significantly more time than the others and to avoid more detailed descriptions.

As time delays can be observed even on the basic level of cellular interactions, we should also expect delays in interpersonal relationships. However, mathematical modeling in sociology and psychology is a rather young branch of applications, and dynamical systems (discrete or continuous) that have been used to describe some aspects of such relationships are typically simple models without delay, cf. e.g. [3,4,5,6,7,12,13,15,16,17,24] in the context of romantic relationships, and also the text-book by Gottman et al. [25] and the article written by Rey [11] focused on the prediction of marital dissolution on the basis of such sort of models. Typically, discrete dynamical systems approach is used in this topic, cf. [4,25]. Continuous dynamical systems in the description of romantic relationships has been introduced by Strogatz [16,26]. The idea of Strogatz influenced many researchers to study different styles of romantic relationships with linear and non-linear influence/ effort terms, cf. [5,7,11,12,13,15,17,24].

It is also worth to point that work by Strogatz also gave birth to a new branch of case study in which famous and well-described historical couples as well as characters from literature and pop-culture undergo analysis of relationship dynamics, cf. e.g. [12,14,27]. In frames of this approach one may find a broadened analysis of Petrarch's platonic feelings toward mistress Laura (true story which took place in 14th century), performed on the basis of facts extracted from historical records, and also characteristics of such fictional couples as Jack Dawson and Rose Calvert (*Titanic* movie), Dan Gallagher and Alexandra "Alex" Forrest (*Fatal Attraction* movie) or Christian de Neuvillette and Roxane (*Cyrano de Bergerac* by Rostand).

Recently, time delays have been introduced to such systems in order to reflect real interpersonal relations better, cf. [28,29,30] with the detailed discussion on introducing delays in [30]. The models presented and studied by Bielczyk et al. [29,30] essentially follow the ideas of Strogatz, because the assumption is for the interaction between partners to be based on linear interdependency with the time delay. On the other hand, linear modeling seems to be a far simplification for the complex problem of dynamics of interpersonal relationships, and nonlinear models appear to be more accurate here, cf. e.g. [5,8,9,10,11,12,13,14,17,25]. The nonlinear model with time delay in the influence terms was proposed and studied by Lion and Ran [28]. In this paper the authors considered the model with a general form of the functions reflecting the influence of the partner's love on the dynamics of each person's emotions and introduce time delays into it. They presented mathematical analysis of the model focusing on the local stability and Hopf bifurcation with respect to the sum of the delays. The analysis was performed under the assumption that, in the absence of any partner, the dynamics of each person is stable. Similar stability analysis can be found in [30], however in that paper all possible styles of romantic relationships were considered.

Among numerous ideas for developing Strogatz approach, there is another promising class of emerging models: some researchers pay attention to conscious activities of partners who moderate actions in a way to obtain an optimal outcome for themselves, from among all the given possibilities. In other words, partners in these models are described as intelligent agents capable of making decisions instead of being driven only by emotions. For example, [5] contributed to the economic theory of addictive behavior in a context of romantic relationships, by analysis of optimal control problem for a dyadic dynamical system. Another example can be found in [11], where it is shown that, whenever partners are similar in terms of emotional attributes, there is an optimal effort strategy leading to a stable and happy coexistence. This approach is interesting especially due to the idea of so called second law of thermodynamics for sentimental relationships which means a tendency for the initial feeling in the relationship to fade away with time if there is no prompt from agents, which must be consciously counteracted by the partners. However, there are certain limitations for this model. First, in this framework the partners are supposed to share all personal traits, which is unlikely to happen in reality. Second, the assumption is for the relationship always to start from a fiery feeling which is often untrue.

1.3. The proposed model

In the paper we follow the ideas of [29] enriched by adding reactivity to each other's appeal to the model, but we also consider a more compound nonlinear model. In both cases we would like to describe system dynamics for partners who emotionally interact with each other, and there is a delay in the interaction involved.

There is a class of relationships we would like to describe, and certain limitations of the model. Primarily, we aim at describing relationships between individuals who are involved in the relationship deep enough and are emphatic enough to be able to involuntarily react to each other's moods. In the analysis, the relationship can turn out to be stable or unstable (fast breaking) and in that sense it not necessarily has to be long-lasting in order to undergo modeling.

Another assumption would be that partners should interact physically and emotionally on a daily basis, which means this is not a long-distance relationship. Since we would like to model mutual impact on emotional states continuous, this regards additional condition which is living together. That is for two major reasons:

(1) interplay of emotional states in partners has to be fluent in order to use ordinary differential equations. If the partners are only dating, two problems arise: first, the media becomes a mediator in the relationship and there are additional delays resulting from information transfer, like time spent to get access to internet and read emails, and second, emotional interactions go along with meetings which are usually irregular in time.
(2) for the couple living apart, it becomes much easier to separate and wait for the other person to emotionally recover whenever they find themselves in a negative state, which spoils the idea of coupled emotional dynamics.

Another assumption is that partners interact both by verbal and non-verbal communication, and also by common activities which have a great impact on the development of the relationship in terms of understanding each other's emotions and emotional proximity [31,32]. This assumption is necessary to justify common value of delay for both partners in the model, it can only result from emotional involvement, deep motivation and training.

The delays in the model do not come from information processing, as opposed to emotional reactivity. We neither mean time necessary for receiving messages through media nor time spent on understanding partner's statements. And as for emotion-driven delays, it was previously mentioned that there is full spectra of primary and compound emotions that have impact on the outcome mood which is then described as one-dimensional variable. Since for each person, regarding circumstances, different emotions are formed with different delays, it is hard to reflect this fact in an analytically treatable system. Thus, we decided to compose two variables: one representing all the primary and "fast" emotions (with very small delay to stimuli that can be approximated by zero delay) and one representing all the delayed compound emotions.

For the delayed case, imposing constant delay simplifies the problem enough to be treatable although one cannot argue that this is a very far simplification. However, this is a first, very important step on the way for the DET approach to develop from numerically to analytically treatable systems. Another question is timescale of delay. We only consider couples who strongly interact and are deeply emotionally involved, however the most appropriate value of delay for a given couple can depend on mandane circumstances such as amount of working hours during the day and if the partners work together or not. It is well known in psychology of emotion that intellectual engagement suppresses moods, thus emotional dynamics of intellectually hard-working couples is naturally slower. In general, timescale for delay in this model is in range between hours and days. If one had to give an example to the ongoing emotional processing, it could be:

"Yesterday I did not want to bother you because you had a hard day at job and I was worrying about you, but today I finally took time to think about your attitude and I have to tell you I very much dislike the way you called my Mum yesterday morning."

"I know we were arguing just yesterday but this afternoon I was biking home from work and suddenly it just came to my mind that despite some small differences between us I am very happy with you."

In general, in this paper we are looking for mathematical description of conditions necessary for this class of relationships to be stable. It is well known that if discrete time delay is introduced to the system in a manner typical for modeling of a natural phenomena, this may lead to destabilization of steady states and cause delayed induced oscillations. In this paper we discuss the possibility that discrete time delay have a stabilizing effect. Although it is known that if any stability switches occur (and model parameters do not depend on time delay), the steady state would be eventually unstable if the delay is sufficiently large (see [33]), but we have in mind social interactions, and therefore time delay should be bounded. For the models presented in this paper there are two stability switches — from instability to stability at some critical delay τ_1 and to instability again for some $\tau_2 > \tau_1$. Hence, we can easily imagine the situation that τ_2 is out of the range of reasonable delays and only the change at τ_1 can appear in reality yielding stabilization. We would like to emphasize that this stabilizing effect is a result of inner dynamics of the system, not the external impact on it. Stabilization due to external influence is well known in the control theory, cf. e.g. [34].

The paper is organized in a sandwich manner. In the next section, we present the family of linear models studied by Bielczyk et al. [30]. Next, we propose more general class of linear models and explain the reason of studying such modification while focusing on one particular type of the model from that class. We also give and discuss an example of a relationship described in such a model. Finally, we present the nonlinear model for which we observe similar stability switches as in the linear model in study.

In the third section we turn to analysis and formulate a general theorem guaranteeing that the steady state for a system of two differential equations can gain stability due to the presence of time delay.

In the last two sections we get back to sociology and focus on application of the proved theorem to the models of romantic relationships proposed in the second section and provide with some numerical simulations. We also include general implications of our account in Conclusions.

2. Romeo and Juliet model

In this paper we essentially follow the ideas of [16,26] where the changes of Romeo's and Juliet's love/hate in time are described as a system of two linear ODEs. As Strogatz, we imply interpersonal dependency in the relationship by relating states of the partners with each other but ignoring the explicit cooperation between partners which has been studied e.g. by [35]. In other words, the analysis is based on the interplay between partners' emotional states and does not concern their conscious or subconscious activities. This dynamics is only driven by reactions, so the partners are not intelligent agents here.

The signs of coefficients in such system specify the romantic styles of lovers. In the similar way Felmlee and Greenberg described dyadic interactions in [6]. However, their system slightly differs from those considered by Strogatz. In the system proposed by [6] some signs of coefficients are fixed. The idea of introducing delay to the model proposed by Strogatz comes from [29,30]. However, the models studied by [29,30] slightly differ from each other. We explain this difference below, while presenting the models.

2.1. Linear model

At first, we present a model of love affairs dynamics proposed by [16,26] and also by [6]. This model reads

$$\begin{cases} \dot{x}(t) = \alpha_1 x(t) + \beta_1 y(t), \\ \dot{y}(t) = \beta_2 x(t) + \alpha_2 y(t), \end{cases} \tag{2.1}$$

where $x(t)$ denotes Romeo's emotions (love if $x(t) > 0$, hate if $x(t) < 0$) for Juliet at time t, while $y(t)$ denotes Juliet's love/hate for Romeo at time t. The coefficients $\alpha_i,\ i = 1,2$, reflect the influence of their own emotions on themselves, while β_i describe the direct effect of their love on the partner. Different signs of the coefficients $\alpha_i, \beta_i,\ i = 1,2$, describe different romantic styles. More precisely, Strogatz claimed that for $\alpha_1, \beta_1 > 0$ Romeo is an eager beaver (meaning that Romeo is encouraged by his own feelings as well as Juliet's feelings for him), for $\alpha_1 > 0, \beta_1 < 0$ he is a narcissist (who wants more of what he feels but retreats from Juliet's feelings), for $\alpha_1 < 0, \beta_1 > 0$ he is a cautious (or secure) lover (that is Romeo retreats from his own feelings but is encouraged by Juliet's one) and for $\alpha_1, \beta_1 < 0$ he is a hermit (that is Romeo retreats from his own feelings as well as Juliet's one), cf. [15].

In the papers [7,24,12,13] the notion of appeal was included to the model. These authors distinguished between reaction to the partner's emotions and partner's appeal, and introduced different variables describing these two kinds of reactivity. Hence, the more general linear model with appeal can be considered

$$\begin{cases} \dot{x}(t) = \alpha_1 x(t) + \beta_1 y(t) + r_1 A_2, \\ \dot{y}(t) = \beta_2 x(t) + \alpha_2 y(t) + r_2 A_1, \end{cases} \tag{2.2}$$

where A_1, A_2 are constant coefficients reflecting the appeal of Romeo and Juliet, respectively and r_1 describes Romeo's reaction to Juliet's appeal and r_2 – reaction of Juliet to Romeo's appeal. It can be easily noticed that the mathematical analysis of the model (2.2) is identical as of (2.1), the only difference is the location of the steady state, that is the point (0,0) for (2.1) and $\left(\frac{\beta_1 r_2 A_1 - \beta_2 r_1 A_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \frac{\alpha_2 r_1 A_2 - \alpha_1 r_2 A_1}{\alpha_1\beta_2 - \alpha_2\beta_1}\right)$ for (2.2) (assuming that $\alpha_1\beta_2 \neq \alpha_2\beta_1$). This extension of the Strogatz model is interesting especially because it enables to derive properties which formally describe the process of falling in love: in frames of this approach it is possible to give an example of a system with partners starting from indifference and tending toward a positive plateau. On the other hand, the appeal may play the crucial role in the socio- and psychological interpretation of the model, and for this reason its impact on evolution of a relationship should not be ignored.

There is also another possible extension of the Strogatz model which is another step toward better description of real relationships evolution. A class of models studied by Bielczyk et al. [30] is also constructed on the basis of Eq. (2.1), assuming that at least one of the reaction terms $\alpha_1 x, \beta_1 y, \alpha_2 y, \beta_2 x$ is delayed, while the others are instantaneous. This class of models reads

$$\begin{cases} \dot{x}(t) = \alpha_1 x(t - \delta_{11}\tau) + \beta_1 y(t - \delta_{12}\tau), \\ \dot{y}(t) = \beta_2 x(t - \delta_{21}\tau) + \alpha_2 y(t - \delta_{22}\tau), \end{cases} \tag{2.3}$$

where $\delta_{ij} = 0$ when the reaction is instantaneous, while $\delta_{ij} = 1$ when it is delayed. These equations are constructed on a basis of the Strogatz model, however $x(t)$ and $y(t)$ variables has been redefined as a "level of satisfaction from the relationship".

This is because satisfaction is a one-dimensional variable, more precise than "happiness" or "level of emotions", and restricted to a part of emotional life referring only to the relationship.

We also associate the desired stability in the relationship with a marriage or other long-term relationship. This means we assume that time scale for stability we are describing is at least several years. This fact does not come out explicit in the analysis though.

The analysis presented in [30] is focused on the possibility of stability switches due to increasing delay. An interesting result is obtained in that paper: multiple stability switches are possible only when at least one of the partners reacts with delay to their own state. As for the other classes of models, where both "self-reactions" are instantaneous, there can appear at most one stability switch. More precisely, if the steady state is stable for $\tau = 0$, then it remains stable for all positive delays or it can lose stability for some threshold delay τ_{cr} and cannot regain it for larger delays, while if it is unstable for $\tau = 0$, then it remains unstable for all positive values of τ.

In this paper, we combine the two possibilities for extension of the Strogatz model introduced in Eqs. (2.2) and (2.3), with $x(t)$ and $y(t)$ interpreted as a level of satisfaction from the relationship. This yields a class of models of a general form

$$\begin{cases} \dot{x}(t) = \alpha_1 x(t) + a x(t-\tau_{xx}) + \beta_1 y(t) + b y(t-\tau_{xy}) + r_1 A_2, \\ \dot{y}(t) = \alpha_2 x(t) + A x(t-\tau_{yx}) + \beta_2 y(t) + B y(t-\tau_{yy}) + r_2 A_1, \end{cases} \tag{2.4}$$

where parameters $\alpha_i, \beta_i, r_1, A_i,\ i = 1,2$, and a, b, A, B can have arbitrary signs (although A_1 and A_2 are supposed to be positive), as in the case of models considered by Strogatz, and delays can have different positive values. People indeed react instantaneously on emotional stimuli, but they can also make corrections to their level of satisfaction in a result of time delayed reconsideration, thus in our opinion such a description better reflects the human nature than a model given by Eq. (2.2).

The model introduced above is too general to analyze though. In order to attribute personality traits to partners and describe a certain situation, at least signum shall be determined for all the coefficients in the model. Furthermore, even though extension of a simple linear model by adding a time delay seems to be only a slight change, there are certain constraints for the model to be analytically treatable afterward, cf. [29,30]. It is hard to pursue calculations if there is more than one nonzero value of delay in the system, thus we shall restrict values of delay to be either 0 or τ.

Even after implementation of these constraints, we still have 3^8 substantially different versions of the model to treat (since each of 8 coefficients in (2.4) may be either equal to zero, positive or negative). Theoretically, it gives a full spectra of models, but some of the combinations of coefficients do not make sense if we are looking for conditions of stability. In an example, at least one of the coefficients must be negative, otherwise the system does not gain stability at all. We also assume at least $\alpha_i, \beta_i,\ i = 1,2$, to be non-zero, because the partners should be reactive and prompt to emotional stimuli such as their own and the partner's temporary state. For a, b, A, B there is no such constraint because an ability to correct someone's own state after a period of time depends on a level of deliberation of the person, and therefore describes a personal trait.

In fact, we are able to analyze every system that fulfills the above requirements. This means we may construct a number of systems by describing a given couple in a language of reactivities to themselves and to each other, which provides a room for further research. In order to present capabilities which appear in a result of this approach, we pursue a full analysis of an exemplary system.

In the story we would like to model, Romeo is in deep love with Juliet, he gets excited by Juliet's love for him, and further is spurred on by his own affectionate feelings for her [citation from 26]. In a result, we assume $\alpha_1,\ \beta_1 > 0$. However, immediate positive reactions of Romeo are reconsidered after some time, and his reactions to Juliet's states are negative in the end (his level of satisfaction drops whenever Juliet is satisfied and the opposite).

On the contrary, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide [again cited from 26], but on the other hand, her own level of satisfaction intensifies itself which means Juliet has an insight into her own state and whenever she realizes herself to be satisfied, it makes her even more satisfied and the opposite, hence $\alpha_2 < 0$ and $\beta_2 > 0$.

Thus, we describe the couple using a system of two ordinary differential equations with time delay

$$\begin{cases} \dot{x}(t) = \alpha_1 x(t) + \beta_1 y(t) - b y(t-\tau) + r_1 A_2, \\ \dot{y}(t) = -\alpha_2 x(t) + \beta_2 y(t) + r_2 A_1, \end{cases} \tag{2.5}$$

where $\alpha_i, \beta_i, r_i,\ i = 1,2, b, A_1$ are positive parameters, while A_2 is supposed to be negative. Notice, that the parameters $\alpha_1, \beta_i,\ i = 1,2$, and b in (2.5) are absolute values of the parameters in (2.4). Initial data $(x(t), y(t)) \in \mathbb{R}^+$ for $t \in [-\tau, 0]$ reflect emotions at the initial time interval.

Looking at classic literature, this model well describes relationship of Martin Eden and Ruth Morse, main characters of a masterpiece "Martin Eden" by London and Eden [36]. The plot concentrates on story of a young mariner, Martin, and a young lady from higher class, Ruth. Martin was truly affectionate with Ruth and impressed by her education and manners, whereas Ruth was taught from childhood to be calculated and self-centered, treated Martin disrespectfully and thought of messing with a man from lower class as a mercy. He was positive about his own and her feelings, and he was concerned about Ruth's level of satisfaction more than about his own, and he also used to think a lot about Ruth's states and not about his own. The more unsatisfied Ruth was in the past, the more positive and motivated to act Martin was becoming. Nevertheless, Martin was not attracted by Ruth's appeal since she was not an attractive woman in a classic sense of the word; she only impressed him with intellect and manners. On the other hand, Ruth felt cooled down anytime Martin was showing her his satisfaction and happiness. She was much concerned about her own satisfaction and did not use to deliberate the past. After Martin decided to educate, and then became a famous writer and a bitter person disappointed by other people at the same time, Ruth's level of satisfaction immediately increased. On the other hand, Martin was very attractive physically, sociable and popular in his mariner environment, and his wild and manly appeal had a great impact on Ruth. Thus, our model can well describe such a situation, especially when $\beta_1 > \alpha_1$ and $\alpha_2 > \beta_2$.

In the following we provide analysis of stability conditions for the above model. As mentioned previously, this is one of numerous possibilities which may be described in frames of the introduced approach, however there is no room for analysis of all the systems in this paper.

2.2. Nonlinear model

As we have explained in Introduction, constructing linear models is not the most relevant approach for the description of interpersonal relationships, since they can be of nonlinear nature. This means linear description has only local character, and is valid only near the steady state.

As previously, there is a bunch of models to examine as soon as we take into account nonlinear terms. From psycho- or sociological point of view, extending the model by terms of a form $x(t)y(t)$ or $x(t-\tau)y(t-\tau)$ is most appropriate because these terms describe reaction to a combination of two states at the same time. It is also worth to consider terms of a form $x(t)x(t)$ and $x(t-\tau)x(t-\tau)$, because they are always of the same signum and may describe an always positive or always negative reaction to a partner, as well as reaction of a person to their own state, which is always positive (self-activating attitude) or always negative (self-abasement attitude).

Here, we also choose an exemplary nonlinear system in which delay has a stabilizing effect. The model reads

$$\begin{cases} \dot{x}(t) = -\alpha_x x(t) + \eta x^2(t) + \beta_x y(t-\tau) - \gamma_x x(t)y(t) + r_1 A_2, \\ \dot{y}(t) = \beta_y x(t) - \alpha_y y(t) + \gamma_y x(t)y(t) + r_2 A_1, \end{cases} \tag{2.6}$$

with positive coefficients.

In this relationship, Romeo reacts negatively to his own states and positively to Juliet's states, although with delay. He is also negative toward their joint state $x(t)y(t)$ which means his level of satisfaction increases whenever one of them feels unsatisfied and decreases whenever they are both satisfied or unsatisfied. This may contribute to a typical manly attitude which makes Romeo feel responsible for the relationship and obliges himself to solve problems like difference in level of satisfaction. We are talking about reactions which do not have to be conscious especially when they are instantaneous as this one. Romeo also has an egoistic feature represented by a factor η, which describes his increase of satisfaction as a response to all his states. It gives more compound view at Romeo's personality which is rather self-centered. It also reflects results of a research by Wood [37] suggesting that for men, relationships are not as central as for women.

On the contrary, Juliet (her emotions are described by the function $y(t)$) reacts positively to similar state of the partner (both positive or both negative), which is described by the term $x(t)y(t)$. This also means she reacts negatively to lack of this conformity between partners which can be described as co-feeling factor. If we assume both Juliet's and Romeo's negative states to be a conflict, this suggests for Juliet to display positive communication attitude in such conditions [38]. At the same time analysis of her own state cools down her level of satisfaction (she reacts with negative coefficient which makes her react positively to her own lack of satisfaction), while she reacts positively on her partner positive state and negatively on his negative state. This can be described as cautious lover attitude, cf. also [29,30].

A good example of such a couple can be Rhett Butler and Scarlett O'Hara, main characters of "Gone with the Wind" by Mitchell [39]. In this world-wide known novel, and also movie, Rhett was a self-centered, but deliberate man whereas Scarlett was an impulsive, but fragile and cautious lover. Analysis of this model can be found in Section 4.

3. General theorem

In this section we present mathematical approach that can be used in analysis of such models as (2.3), (2.5), and (2.6). We prove that in the cases we study there is a range of reasonable values of delay that lead to stabilization of the system.

Typically, considering local stability of a steady state it is enough to study a linearized system and the stability of the zero steady state which yields to the characteristic function (a quasi-polynomial in this case). For any system of two ordinary differential equations with one discrete delay its characteristic function has a general form

$$W(\lambda) = \lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau} + ce^{-2\lambda\tau}, \tag{3.1}$$

where a_0, a_1, b_0, b_1, c are arbitrary constants. To simplify the analysis we assume $c = 0$ in this paper. However, due to the continuous dependence of eigenvalues on the model parameters, cf. e.g. [40], the results are also valid for sufficiently small values of c. In this section by the steady state we always understand the zero steady state of a linear (linearized) system. By stability switch we understand the change of stability of the steady state from stable to unstable or reverse. For nonlinear systems the occurrence of stability switch is typically associated with Hopf bifurcation, which means the appearance of periodic orbits that can be stable or unstable, depending on the type of bifurcation.

One of the mathematical tools that can be used to prove stability or instability is the Mikhailov criterion, cf. e.g. [41].

Lemma 1 (Mikhailov criterion). *Assume that W has no pure imaginary roots. Then, the steady state of the system with the characteristic equation of the form (3.1) is locally stable iff*

$$\Delta_{0\leqslant\omega<\infty} \arg W(i\omega) = \pi.$$

Lemma 1 implies that to study stability it is enough to calculate the total change of the argument of the vector $W(i\omega)$ when omega increases from 0 to ∞. In more general case, when the characteristic function has the form $W(\lambda) = P(\lambda) + Q(\lambda)$ with P being a polynomial of the n-th degree and Q is a linear combination of functions $\lambda^k \exp(-\lambda\tau_k),\ k = 0,\dots,n-1$, the condition guaranteeing local stability reads

$$\Delta_{0\leqslant\omega<\infty} \arg W(i\omega) = \frac{\pi}{2}\deg P = \frac{n\pi}{2}.$$

Before discussing a stabilizing effect of time delay, we give a short review of the cases in which the change of stability cannot occur or time delay destabilize the steady state. Notice that due to the continuous dependence of the roots of $W(\lambda)$ on the coefficients, for the occurrence of a stability switch it is necessary that a pair of purely imaginary roots $\pm i\omega$ of W appears. Thus, there should exist a positive ω such that

$$W(i\omega) = 0 \Rightarrow \left|(i\omega)^2 + a_1 i\omega + a_0\right|^2 = |b_1 i\omega + b_0|^2.$$

Substituting ω by a $\sqrt{x}$ we easily get

$$F(x) = x^2 + (a_1^2 - 2a_0 - b_1^2)x + a_0^2 - b_0^2 = 0.$$

Stability switches can occur only if there exists at least one positive root $\bar{x}$ of F. Moreover, the sign of the derivative of F at points $\bar{x}$ determines the direction of movement of the roots of W in the complex plane with increasing delay, cf. [33]. This means that if $F\prime(\bar{x}) > 0$, then the respective pair of roots of W crosses the imaginary axis from the left to the right-hand complex half-plane, while if $F'(\bar{x}) < 0$ from the right to the left-hand side. Therefore, stability might be gained only if $F'(\bar{x}) < 0$.

Lemma 2. *If $|a_0| < |b_0|$, then*

- *if the steady state is unstable for $\tau = 0$ (i.e. either $a_0 + b_0 < 0$ or $a_1 + b_1 > 0$), then it remains unstable for every $\tau > 0$;*
- *if the steady state is stable for $\tau = 0$ (i.e. $a_0 + b_0 > 0$ and $a_1 + b_1 < 0$), then there exists $\tau_{cr} > 0$ such that the steady state is stable for $0 \leqslant \tau < \tau_{cr}$ and is unstable for $\tau > \tau_{cr}$, implying Hopf bifurcation at $\tau = \tau_{cr}$ if the system is nonlinear.*

Proof. The condition $|a_0| < |b_0|$ implies that $F(0) < 0$, thus F has a unique positive root x_0 and $F\prime(x_0) > 0$. Therefore, the unstable steady state cannot gain stability and the stable steady state loses stability at the first critical value of delay for which the pair of purely imaginary eigenvalues exists and cannot gain it again. For nonlinear systems these conditions are sufficient for Hopf bifurcation. □

Lemma 3. *If $(a_1^2 - 2a_0 - b_1^2)^2 < 4(a_0^2 - b_0^2)$ or $a_1^2 > 2a_0 + b_1^2$ and $|a_0| > |b_0|$, then the stability switches cannot occur.*

Proof. It is easy to see that the assumptions imply that F has no real positive roots where the stability switch can occur. □

Corollary 1. *Stability switches may occur only if $|a_0| > |b_0|$ (i.e. $F(0) > 0$), $a_1^2 < 2a_0 + b_1^2$ and $(a_1^2 - 2a_0 - b_1^2)^2 > 4(a_0^2 - b_0^2)$.*

Proof. Stability switches may occur only if there exist two positive roots of F. This function is a quadratic polynomial of x, and therefore the existence of two real positive roots is equivalent to the condition $a_0^2 - b_0^2 > 0,\ a_1^2 - 2a_0 - b_1^2 < 0$, $(a_1^2 - 2a_0 - b_1^2)^2 - 4(a_0^2 - b_0^2) > 0$. □

Lemma 4. *If for $\tau = 0$ the steady state is a saddle point, then the stability switches cannot occur.*

Proof. In this case we have $a_0 + b_0 < 0$. We use the Mikhalov Criterion stated in Lemma 1 to prove instability of the steady state for all $\tau \geqslant 0$. Let us denote

$$\begin{aligned} W_r(\omega) &= \mathrm{Re}(W(i\omega)) = -\omega^2 + a_0 + b_0\cos(\omega\tau) + b_1\omega\sin(\omega\tau), \\ W_i(\omega) &= \mathrm{Im}(W(i\omega)) = a_1\omega - b_0\sin(\omega\tau) + b_1\omega\cos(\omega\tau). \end{aligned} \tag{3.2}$$

Using the Mikhailov Criterion we study the change of argument of the vector $W(i\omega)$ for ω increasing from 0 to $+\infty$. We see that $W_r(0) = a_0 + b_0 < 0$ and $W_i(0) = 0$, that is $\arg W(0) = \pi$ and

$$\sin \arg W(i\omega) = \frac{a_1\omega - b_0 \sin(\omega\tau) + b_1\omega\cos(\omega\tau)}{\sqrt{W_r^2 + W_i^2}} \to 0,$$
$$\cos \arg W(i\omega) = \frac{-\omega^2 + a_0 + b_0\cos(\omega\tau) + b_1\omega\sin(\omega\tau)}{\sqrt{W_r^2 + W_i^2}} \to -1$$

as $\omega \to +\infty$. Therefore, $\arg W(i\omega) \to \pi$ and the total change is equal to $\pi + 2l\pi - \pi = 2l\pi \neq \pi,\ l \in \mathbb{Z}$, which implies that the steady state is unstable. □

Let the assumptions of Corollary 1 be fulfilled and denote by $0 < x_0 < x_1$ zeros of F with $\omega_i = \sqrt{x_j}, j = 0, 1$. We have $W(i\omega_j) = 0$ for $j = 0, 1$, and this yields

$$\sin(\omega_j\tau_j) = \omega_j \frac{b_1\omega_j^2 + a_1b_0 - a_0b_1}{b_1\omega_j^2 + b_0^2}, \quad \cos(\omega_j\tau_j) = \frac{(b_0 - a_1b_1)\omega_j^2 - a_0b_0}{b_1\omega_j^2 + b_0^2} \quad \text{for} \quad j = 0, 1.$$

This system has unique solutions $\tau_{j0} \in [0, 2\pi/\omega_j)$ for $j = 0, 1$. Then we have a sequence of critical values $(\tau_{jn})_{n\in\mathbb{N}}$ such that

$$\tau_{jn} = \tau_{j0} + \frac{2n\pi}{\omega_j}, \quad j = 0, 1 \quad \text{and} \quad n \in \mathbb{N}.$$

The inequality $\omega_1 > \omega_0$ implies that roots of $W(\lambda)$ cross imaginary axes more often from the left to the right-hand complex half-plane than in the opposite direction. Thus, eventually the steady state would be unstable for large τ. Stability switches may occur only in two cases

1. the steady state is stable for $\tau = 0$ and $\tau_{10} < \tau_{00} < \tau_{11}$ (the first inequality is always fulfilled in this case) or
2. the steady state is an unstable node or unstable focus for $\tau = 0$ and $\tau_{00} < \tau_{10}$.

Now we formulate a theorem for a particular case when the second possibility occurs.

Theorem 1. *Assume that*

$$a_1 < 0, \quad |a_0| > |b_0|, \quad a_0 + b_0 > 0, \quad a_1^2 < 2a_0\left(1 - \sqrt{1 - \left(\frac{b_0}{a_0}\right)^2}\right). \tag{3.3}$$

Then

(i) if $b_0 < 0$ and $a_1^2 < 8|b_0|/9$, then for any $a_0 \in \left(-b_0, 8(b_0/(3a_1))^2\right)$ and b_1 sufficiently close to 0 stability switches occur and the first switch is for $\tau_c < \frac{3|a_1|}{2|b_0|}$;

(ii) if $b_0 > 0$ and $|a_1| < 8\sqrt{a_0}/(9\pi^2)$, then for any $b_0 \in (|a_1|\sqrt{a_0}, 8a_0/(9\pi))$ and b_1 sufficiently close to 0 stability switches occur and the first switch is for $\tau_c < 3\pi/(2\sqrt{a_0})$.

Proof. We present only the idea of the proof here. The full proof is included as Appendix A.

Conditions (3.3) imply that for $\tau = 0$ the steady state is unstable and the assumptions of Corollary 1 hold for sufficiently small b_1. Thus, stability switches may occur. We show that the steady state is locally asymptotically stable for some $\bar{\tau} > 0$. Notice also, that $|a_0| > |b_0|$ together with $a_0 + b_0 > 0$ imply $a_0 > 0$.

Due to continuous dependence of roots of the characteristic function W on its parameters it is enough to prove the theorem for $b_1 = 0$. As in the proof of Lemma 4 we use the Mikhalov criterion. We see that $\arg W(0) = 0$ and Expressions (3.2) imply that the total change of the argument $\arg W(i\omega)$ is equal to $\pi + 2l\pi, l \in \mathbb{Z}$. We show that W_r is decreasing. In the case (i) we find $\bar{\omega}$ such that $W_i(\omega) > 0$ for all $\omega \in (0, \bar{\omega})$ as well as $W_r(\bar{\omega}) < 0$. This yields that the change of $\arg W(i\omega)$ for $\omega \in (0, +\infty)$ is π, so the steady state is stable due to the Mikhalov criterion. In the case (ii) we show that there exits $\bar{\omega}$ such that $W_r(\bar{\omega}) = 0$ and $W_i(\bar{\omega}) > 0$. This, together with the monotonicity of W_r, yields that the change of $\arg W(i\omega)$ for $\omega \in (0, +\infty)$ is π, so the steady state is stable due to the Mikhalov Criterion. The sketch of the Mikhailov hodographs, that is the curves that are drawn by the vector $W(i\omega)$ when ω increases from 0 to ∞, are presented in Fig. 1. □

4. Application of general theorem

Now, we apply General Theorem proved in the previous section to show stability switches for the systems we are interested in, that mean Eqs. (2.5) and (2.6), and particular sets of parameter values. In the case of model (2.5) we have an unstable steady state for delays equal to and near 0, this steady state gains stability at the first threshold value of delay and loses it at the next critical value.

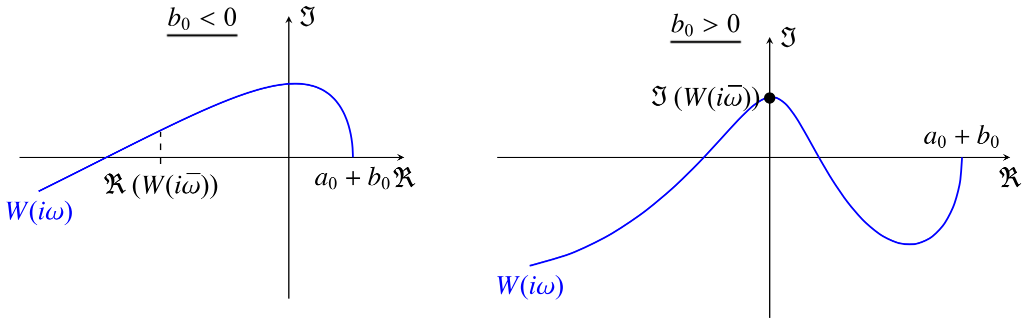

Fig. 1. The sketch of Mikhailov hodographs in the case (i) and (ii) in the left and right-hand graphs, respectively.

The characteristic function for Eqs. (2.5) has the following form

$$W(\lambda) = \lambda^2 - (\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2 + \beta_1\beta_2 - \beta_2 b\exp(-\lambda\tau)$$

and choosing parameters values as

$$\alpha_1 = 0.2, \quad \beta_1 = 1.4, \quad b = 1.25, \quad \alpha_2 = 0.2, \quad \beta_2 = 0.4, \quad r_1 = 2, \quad r_2 = 2, \quad A_1 = 1, \quad A_2 = -1 \tag{4.1}$$

we get $a_1 = -(\alpha_1 + \alpha_2) = -0.4$, $a_0 = \alpha_1\alpha_2 + \beta_1\beta_2 = 0.6$, $b_0 = -\beta_2 b = -0.5, b_1 = 0$ and the steady state is $(x, y) = (7, 4)$. Note that in linear model appeal terms do not affect the stability of the steady state but only the value of the steady state. From mathematical point of view it is not surprising as appeal is modeled as a source term. This allows to move from $(0, 0)$ point and start love affair, but it cannot affect the stability of the steady state, at least in the linear model. This means that reactions on his/her and on partners' feelings determines the stability of the steady state and the appeal terms affect only the position of the steady state. It is an interesting note since it seems as far as we analyze simply structured relationships that can be described on the basis of linear interplay of actions-reactions, appeal does not have impact on stability of the relationship. It can contribute to recent experimental findings on the impact of physical attractiveness on marriage quality [42] which suggest that level of attractiveness usually affects only the initiation of the relationship, as is not predictive in terms of marriage duration (stability).

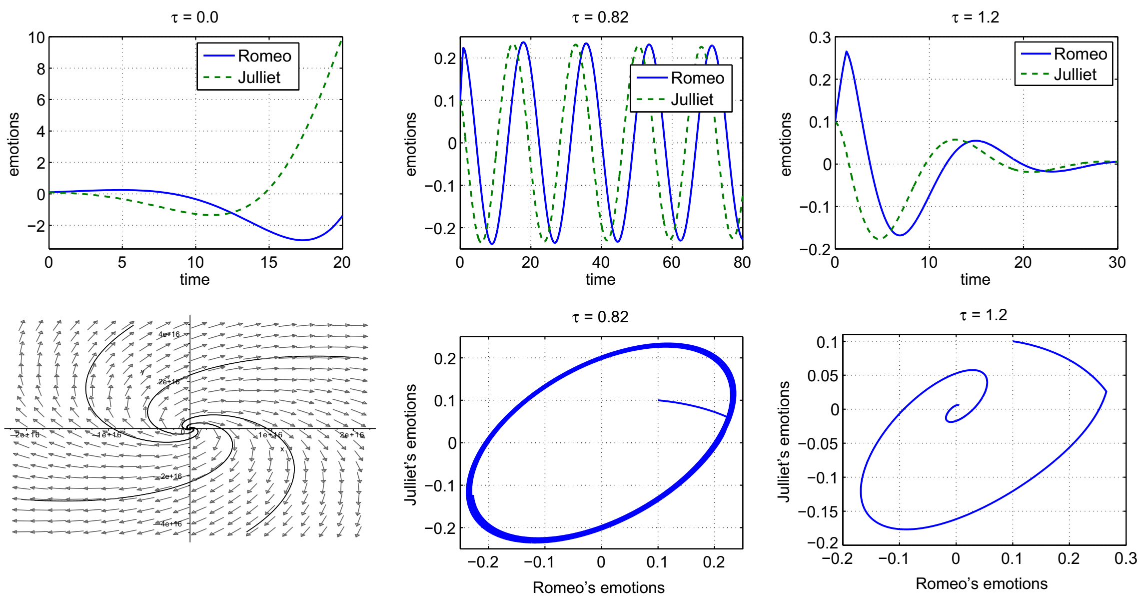

Fig. 2. Solutions to system (2.5) with parameters given in (4.1) for different values of time delay, below and around the first critical value τ_1. The phase portrait of system (2.5) for $\tau = 0$ is presented on the lower left hand-side picture. The Romeo's emotion was shifted by 7 and Juliet's one by 4 to display better the behaviour of the solutions around the steady state $(7, 4)$. We may observe the relationship to be unstable below a certain value of delay. Above this value, it stabilizes. This means some minimum level of sloth in Romeo is beneficial for the relationship.

But the key feature of a linear system with such parameters we may observe is that if R's reaction is delayed for a very short time, the relationship is unstable. There exists a certain value of delay above which the relationship stabilizes which means some minimum level of sloth in R is beneficial for the relationship. However, if delay is big enough, the relationship destabilizes again and for all times, which means too much sloth to be destructive to the relationship as well. In Figs. 2 and 3 the behavior of the solution to (2.5) for different values of time delay is presented. For $\tau = 0$ (see the upper left hand-side picture for graphs of the solutions and the lower left hand-side picture for the "phase portrait", meaning the dependence between the values of R and J variables in the paper) the steady state is unstable and the solutions oscillate with increasing amplitude.

It can be easily checked that the assumptions of Theorem 1 are fulfilled. In this case there exist two critical values of time delay: $\tau_1 \approx 0.81$ and $\tau_2 \approx 2.36$. For τ_1 the steady state gains stability. For $\tau < \tau_1$ the amplitude of oscillation increases with time while for $\tau > \tau_1$ oscillations are dumping. For τ_2 the steady state loses stability. The amplitude of oscillations increases with time for $\tau > \tau_2$. For positive $\tau < \tau_1$ the amplitude of oscillations is growing slower and slower, eventually oscillations become stable, and then start to be dumping as the steady state gains stability (in the middle graphs of Fig. 2 this situation is illustrated for $\tau = 0.82$ and in the right hand-side graphs of Fig. 2 for $\tau = 1.2$). When τ increases further, the convergence to the steady state is faster and oscillations disappear. Oscillations appear again when τ approaches τ_2 (see the left-hand side and middle graphs of Fig. 3 for illustration of this situation for $\tau = 2.2$ and 2.363, respectively) and the steady state becomes unstable again and remains unstable for all $\tau > \tau_2$ (see the right hand-side of Fig. 3 for illustration of the situation for $\tau = 2.6$).

For the nonlinear model (2.6) the dynamic is more diverse. The system has in general up to three steady states. If the appeal terms are equal to 0 we may deduce that if $\Delta = (\gamma_x\beta_y - \alpha_x\gamma_y - \eta\alpha_y)^2 - 4\eta\gamma_x(\alpha_x\alpha_y - \beta_x\beta_y) > 0$, then there exist three steady states: $(0,0), (\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$. On the other hand, if $\Delta < 0$, there exists only one steady state $(0,0)$. It can be noticed, that the trivial steady state is locally asymptotically stable for $\tau = 0$ if $\alpha_x\alpha_y > \beta_x\beta_y$. In this case the steady state remains stable for all $\tau > 0$ (because the auxiliary function $F(\omega) = \omega^4 + (\alpha_x^2 + \alpha_y^2)\omega^2 + (\alpha_x\alpha_y)^2 - (\beta_x\beta_y)^2$ has no positive roots). If the inequality is reverse, then the trivial steady state is a saddle point for $\tau = 0$ and due to Lemma 4 it remains unstable for all $\tau > 0$. However, for non-zero appeal terms the situation is much more complex since in general we have to find roots of a cubic polynomial to calculate the steady state.

To illustrate the dynamics of this model we have chosen the following set of parameters:

$$\begin{aligned} &\alpha_x = 1, \quad \alpha_y = 2, \quad \beta_x = 0.5, \quad \beta_y = 0.2, \quad \gamma_x = 1.26, \quad \gamma_y = 0.4, \\ &\eta = 1.28, \quad r_1 = 0.92, \quad r_2 = 1, \quad A_1 = 2, \quad A_2 = 1. \end{aligned} \tag{4.2}$$

In this case we have only one positive steady state $A = (2,2)$. Note that in the nonlinear models the appeal terms may not only influence the value of steady state, but also the existence and stability of steady states. It may contribute to a common belief that complicated relationships, based on various forms and levels of interactions, correlate with high level of attraction

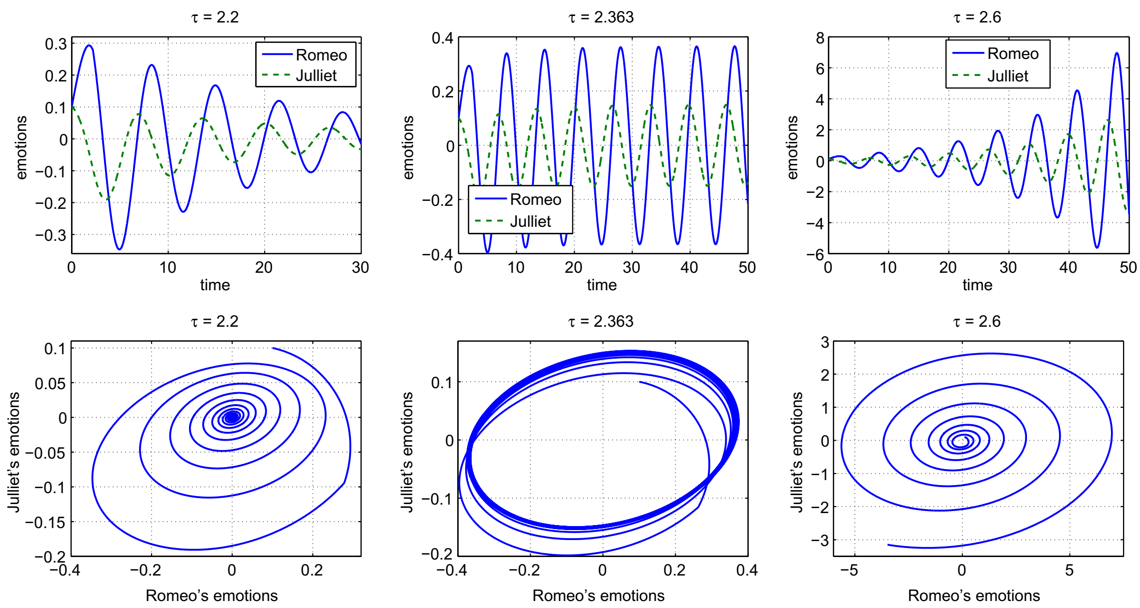

Fig. 3. Solutions to system (2.5) with parameters given in (4.1) for different values of time delay around the second critical value τ_2. The Romeo's emotion was shifted by 7 and Juliet's one by 4 to display better the behaviour of the solutions around the steady state $(7,4)$. The relationship of R and J destabilizes again above some value of delay. This can be interpreted as a finite section of values of delay that can be beneficial for the relationship.

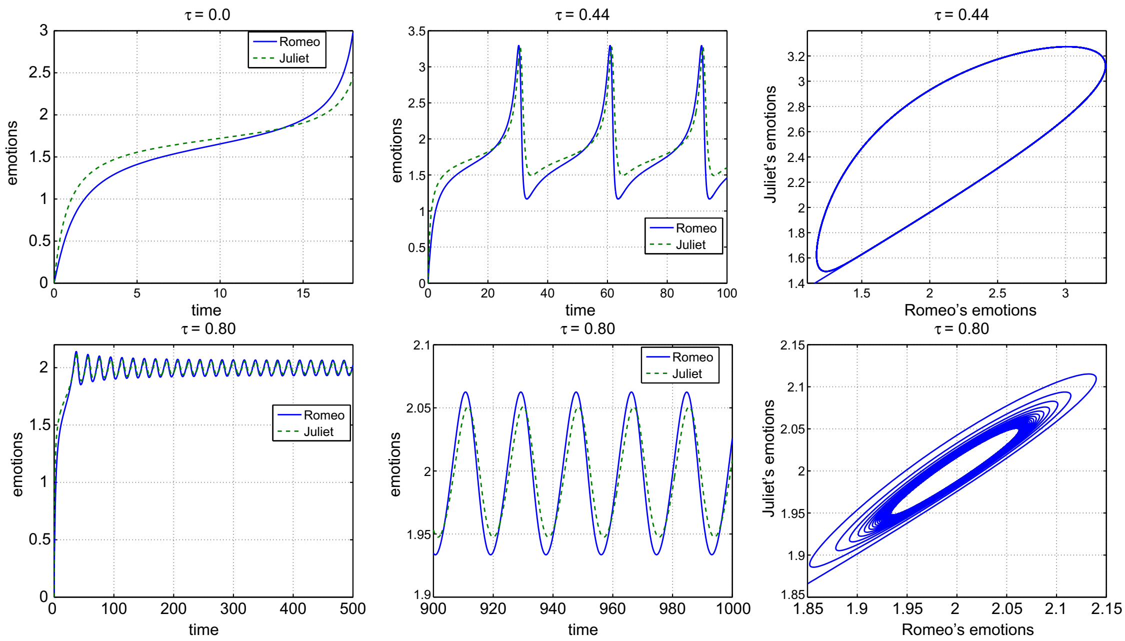

Fig. 4. Solution to (2.6) with initial data $x(0) = 0,\ y(0) = 0$ and for different values of time delay.

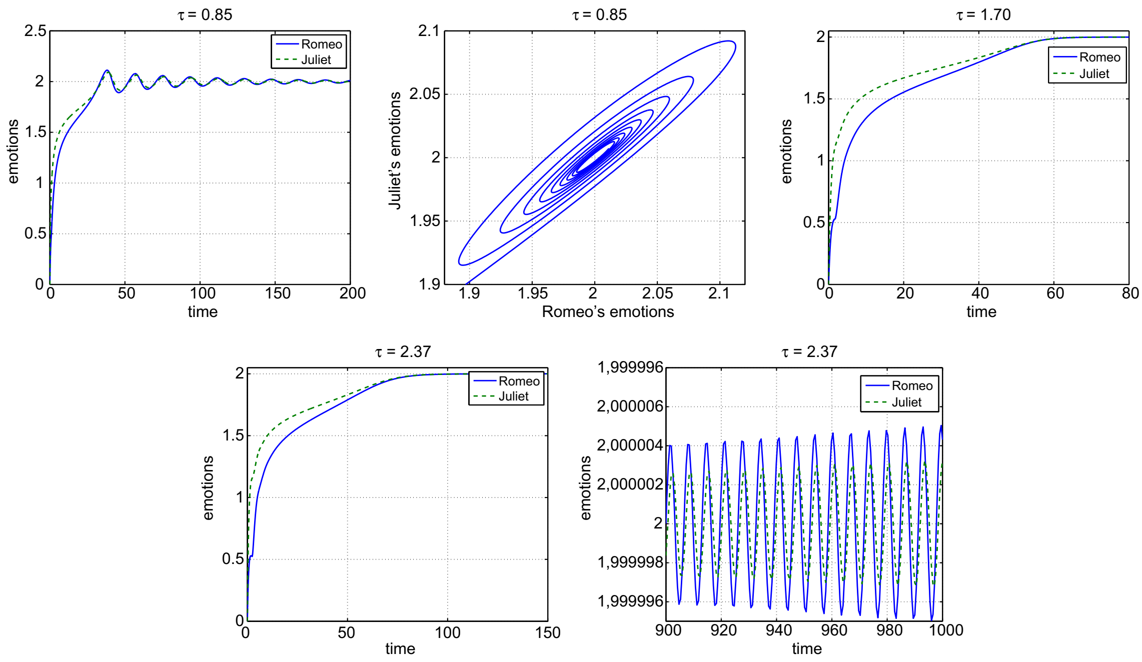

Fig. 5. Solution to (2.6) with initial data $x(0) = 0,\ y(0) = 0$ and for different values of time delay.

of the partners toward each other. Both of these conditions could be associated with high level of hormones which affect both instinctive reactions as attraction and subjective perception as satisfaction.

We may observe that depending on the value of delay we may obtain lack of stability, convergence to the steady state or oscillatory behavior. Exemplary simulations are shown in Figs. 4,5,6.

Linearizing system (2.6) around the steady state C leads to a linear system

$$\begin{cases} \dot{x}(t) = x(t) - 1.2y(t) + 0.5y(t-\tau), \\ \dot{y}(t) = x(t) - 0.6y(t). \end{cases}$$

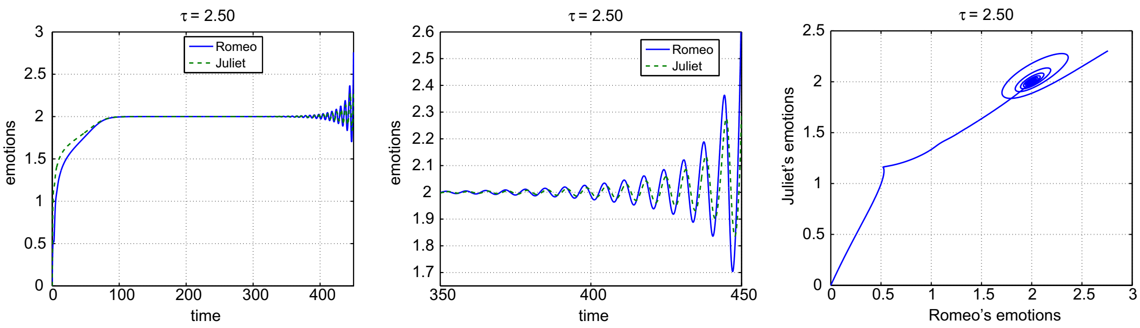

Fig. 6. Solution to (2.6) with initial data $x(0) = 0,\ y(0) = 0$ and for different values of time delay.

Thus, the characteristic function reads

$$W(\lambda) = \lambda^2 + 0.4\lambda + 0.6 - 0.5e^{-\lambda\tau}.$$

It can be easily checked that the assumptions of Theorem 1 are fulfilled. In this case there exist two critical values of time delay: $\tau_1 \approx 0.81$ and $\tau_2 \approx 2.36$. For τ_1 the steady state gains stability. The Hopf bifurcation occurs at this point and is subcritical. For τ_2 the steady state loses stability and the Hopf bifurcation occurs again. The stability switches that occur are presented in Figs. 4,5,6. We see that for small τ solutions to (2.6) tend to ∞ (see the upper left-hand side picture of 4). For lager τ solutions begin to oscillate around the steady state $(2,2)$ (see the upper middle picture of Fig. 4 for the graph of exemplary solutions and the upper right-hand side picture of Fig. 4 for the phase portrait). When time delay parameter approaches the first critical value τ_1, the amplitude of oscillations decreases (see lower pictures of Fig. 4, the right-hand side picture is a phase portrait for $\tau = 0.8$). Next, if $\tau > \tau_1$, solutions converge to $(2,2)$ (see the upper pictures of Fig. 5). When τ is closer to the second critical threshold of τ, namely τ_2, oscillations around $(2,2)$ appear again (see the lower pictures of Fig. 5). For $\tau > \tau_2$ the amplitude of oscillations increases with increasing time. The solutions go to ∞, eventually (compare Fig. 6).

5. Conclusions

In the paper we introduced a novel idea of incorporating delay into classical model of Strogatz [16,26], and we studied two models of interpersonal relationships with time delay. The first model is linear and essentially follows the idea of [16,26], while the latter is nonlinear and more complex in its form.

We attempt to model relationships in a way to obtain dynamics consistent with intuition and experience from real life, however statement that these are only examples from a broad range of possibilities, is true. We hope this can be an inspiration for a further research and exploitation of possibilities provided by Eq. (2.4).

In both cases, we assume for one of the partners to react instantaneously to the other partner but reconsider his/her immediate reaction after some time (and the other partner is instant is their reactions). This is a example and many other configurations of delayed and instant terms are possible.

We have shown that considerable time delays involved in such models can bring stability to an unstable steady state which means the systems unstable without time delay can gain stability for certain range of delays. This is an important result since it mathematically justifies a range of intuitive phenomenas in social psychology (or psychology of relationships).

First, it shows that even positive emotional reactivity can lead to destabilization of the relationship in some cases, thus reflecting each other's moods (personal traits expressed by positive model coefficients) is not a sufficient condition for the relationship to be stable. This is consistent with a common situation when two persons of a great empathy get together, but for some vague reasons they cannot get on well with each other. Our results show it is not only important to react to each other, but also to keep a certain range of time necessary for some of compound emotions to form.

Second, it shows that making corrections to the level of deliberation in the relationship without even trying to change personal traits in partners (which is hard and time costful in real therapy) can cause it gains stability. Mind that this mathematically justifies the sense of working on communication in the relationship. With time passing by, most of successful couples learns each other's emotions and works out reactivity patterns, with emphasis on the time of reaction, and here we suggest explanation for why this can really improve the relationship stability.

Third, it shows how the relationship can arise from a long-term acquaintance or friendship, where future partners have a contact and work out the communication patterns. This is because from extending the simple model with delay of a form (2.3) with appeal terms is a new feature of the system, which may hence start from indifference of both partners and reach a nonzero steady point which well describes a development of the relationship.

All these three conclusions from our account are also consistent with the common belief that in real life immediate emotional reactions can lead to unwanted effects and it is beneficial to reconsider situation in order to make correction to one's level of satisfaction, cf. [43]. However, ability to take time for deliberation is a personal trait. In the analysis, it comes out that

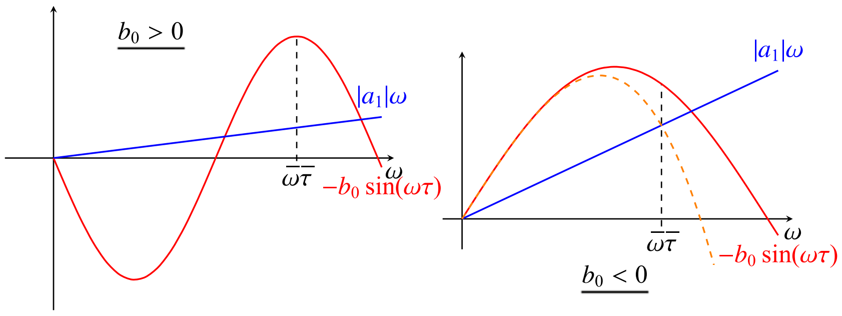

Fig. 7. The idea of choosing $\bar{\omega}$.

time consumed for the deliberation cannot be too long in order to bring good effects. It is consistent with daily experience since overdoing deliberation can lead to breakup.

Last but not the least, our findings are also consistent with statement of classical psychology, in which relationship dynamics is described as eternal trade-off between approach and avoidance as a natural mechanism for shaping interpersonal relations, including romantic feelings, cf. [44].

Acknowledgments

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Appendix A

Here we present the full proof of Theorem 1.

Proof. In the proof we assume $b_1 = 0$. In Fig. 7 the idea of choosing $\bar{\omega}$ in both cases is illustrated.

The case (i). Let us recall that $b_0 < 0$ in this case. Define

$$\bar{\tau} = \frac{3|a_1|}{2|b_0|} \quad \text{and} \quad \bar{\omega} = \frac{\sqrt{8}}{3}\frac{|b_0|}{|a_1|}.$$

For $\tau = \bar{\tau}$, we have

$$W_r'(\omega) = -2\omega + |b_0|\bar{\tau}\sin(\omega\bar{\tau}) \leqslant \omega(\bar{\tau}^2|b_0| - 2) = \omega\left(\frac{9a_1^2}{4|b_0|} - 2\right) < 0$$

since $\sin(\omega\bar{\tau}) \leqslant \omega\bar{\tau}$ and $a_1^2 < 8|b_0|/9$. Notice that

$$W_i'(0) = -|a_1| + |b_0|\tau > 0, \quad \text{for} \quad \tau > \frac{|a_1|}{|b_0|} = \frac{a_1}{b_0}.$$

This means that for $\tau = \bar{\tau}$ the equality $W_i(\omega) > 0$ holds for some open interval $(0, \hat{\omega})$, where $\hat{\omega} > 0$ depends on $\bar{\tau}$.

Now, we estimate $W_i(\omega)$ in the following way

$$W_i(\omega) = -|a_1|\omega + |b_0|\sin(\omega\tau) \geqslant -|a_1|\omega + |b_0|\left(\omega\tau - \frac{\omega^3\tau^3}{6}\right),$$

independently of τ. Hence,

$$W_i(\bar{\omega}) \geqslant \bar{\omega}\left(|b_0|\bar{\tau} - |a_1| - \frac{|b_0|\bar{\tau}^3\bar{\omega}^2}{6}\right).$$

It can be easily seen that $W_i(\omega) > 0$ for $\omega \in (0, \bar{\omega})$ since $\bar{\omega}^2 = (|b_0|\bar{\tau} - |a_1|)\frac{6}{|b_0|\bar{\tau}^3}$.

It remains to prove that $W_r(\bar{\omega}) < 0$. In fact, a simple calculation for $\tau = \bar{\tau}$ yields

$$W_r(\bar{\omega}) = -\bar{\omega}^2 + a_0 + b_0\cos(\bar{\omega}\bar{\tau}) = -\frac{8b_0^2}{9a_1^2} + a_0 - |b_0|\cos\left(\sqrt{2}\right) < -\frac{8b_0^2}{9a_1^2} + a_0 < 0$$

due to the assumptions.

The case (ii). Let us recall that $b_0 > 0$ in this case. Define

$$\bar{\tau} = \frac{3\pi}{2\sqrt{a_0}} \quad \text{and} \quad \bar{\omega} = \sqrt{a_0},$$

which implies $\bar{\omega}\bar{\tau} = \frac{3\pi}{2}$.

The function cosine is decreasing on $(0, \pi)$, and therefore for $\omega < \pi/\bar{\tau}$ the function W_r for $\tau = \bar{\tau}$ is decreasing as a sum of two decreasing functions. We also have

$$W_r'(\omega) = -2\omega + b_0\tau\sin(\omega\tau) \leqslant -2\omega + b_0\tau$$

for any τ. This implies that for $\omega > b_0\tau/2$ we have $W_r'(\omega) < 0$. Thus, for $\tau = \bar{\tau}$ we have W_r decreasing on $(0, \frac{\pi}{\bar{\tau}})$ and on $\left(\frac{b_0\bar{\tau}}{2}, \infty\right)$. However,

$$\frac{b_0\bar{\tau}}{2} < \frac{\pi}{\bar{\tau}} \quad \text{since} \quad b_0 < \frac{2\pi}{\bar{\tau}^2} = \frac{8a_0}{9\pi}.$$

Thus, for $\tau = \bar{\tau}$ the function $W_r'(\omega)$ is decreasing for all $\omega > 0$. Simple calculation yield

$$W_i(\bar{\omega}) = -|a_1|\sqrt{a_0} + b_0 > 0 \quad \text{and} \quad W_r(\bar{\omega}) = -a_0 + a_0 = 0,$$

due to the fact that $\bar{\omega}\bar{\tau} = 3\pi/2$. $\square$

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